

On the high-temperature behaviour of the closed superstring

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Abstract: *The high-temperature expansion for closed super-string one-loop free energy is studied. The Laurent series representation is obtained and its sum is analytically continued in order to investigate the nature of the critical (Hagedorn) temperature. It is found that beyond this critical temperature the statistical sum contribution of the free energy is finite but has an imaginary part, signalling a possible metastability of the system.*

PACS number(s): 11.17 Theories of strings and other extended objects

The interest in (super)-strings at non-zero temperature has been recently grown (see for example Refs. [1-6]). One of the main reasons for these investigations is related to the thermodynamics of the early universe (see [7] and references therein) as well as to the attempts to make use of strings for the description of the high-temperature limit of the confining phase of large- N $SU(N)$ Yang-Mills theory [8,9].

Since the early days of dual string models it is known that such extended objects have an exponential dependence on the mass of the asymptotic state level density. This leads to the existence of the Hagedorn temperature [1] and the breakdown above this critical temperature of the correspondence between canonical microcanonical ensemble. We also mention that the Hagedorn temperature is thought to be the critical temperature for a first order phase transition with a large latent heat [3].

In a series of recent papers [11,12], making use of the so-called Mellin-Barnes representation for the one-loop open superstring free energy, the high temperature behaviour for open and closed bosonic string and open superstring has been investigated. In particular, the Hagedorn temperatures have been characterized as radii of convergence of the associated Laurent series representations. The aim of this note is to complete the analysis to the closed superstring, which, besides to be free from infrared divergences, which plague the bosonic strings, presents some peculiarity. The most important one being the existence of the free energy at the critical temperature, in contrast to the open cases, where the free energy has a pole singularity. As a consequence, the derivation of the related Laurent representation and its analytical continuation seems worth being investigated.

To begin with, we recall that there exist several representations for the one-loop string free energy. One of these representations [6] gives a modular-invariant expression for the free energy. However, this and all the other well known representations [5] are *integral* ones in which the Hagedorn temperature appears as the convergence condition in the ultraviolet limit of a certain integral. Here we shall make use of the well-known approach to finite temperature field theory and then we shall generalize it to the superstring case.

We recall that if we are dealing with bosons (fermions) at finite temperature (β being the inverse of the temperature) in a D -dimensional space-time, we may consider the fields on

$S^1 \times M^{D-1}$, the imaginary time variable $\tau = it$ being compactified with boson (fermion) fields assumed to be periodic (anti-periodic) in β . The related one-loop partition function may be written (we consider scalar field ϕ , the spinor case can be treated similarly)

$$Z_\beta = \int [d\phi] e^{-\frac{1}{2} \int_0^\beta d\tau \int dx \phi L_\beta \phi}, \quad (1)$$

where $L_\beta = -\partial_\tau^2 + \mathbf{p}^2 + m^2$. The related free energy reads

$$\beta \mathcal{F}(\beta) = -\log Z_\beta = \frac{1}{2} \log \det L_\beta = -\frac{1}{2} \int_0^\infty dt t^{\varepsilon-1} \text{Tr} e^{-tL_\beta}. \quad (2)$$

In the above equation, the determinant of the differential operator L_β has been regularized by introducing the ultraviolet cut-off ε . For scalar fields we have

$$\text{Tr}_b e^{-tL_\beta} = \frac{V_{D-1}}{(4\pi t)^{\frac{D-1}{2}}} \sum_{\mathbb{Z}} e^{-\frac{4\pi^2 n^2 t}{\beta^2}} e^{-tm^2}. \quad (3)$$

The corresponding fermionic contribution reads

$$\text{Tr}_f e^{-tL_\beta} = \frac{V_{D-1}}{(4\pi t)^{\frac{D-1}{2}}} \sum_{\mathbb{Z}} e^{-\frac{\pi^2 (2n+1)^2 t}{\beta^2}} e^{-tm^2}. \quad (4)$$

Adding the two terms, using the Poisson resummation formula and separating the vacuum term ($n = 0$), which does not depend on β , one arrives at the following (supersymmetric) statistical sum contribution to the free energy density ($D = 10$)

$$F(\beta) = F_b(\beta) + F_f(\beta) = -\frac{1}{2} \int_0^\infty dt t^{\varepsilon-1} (4\pi t)^{-5} e^{-tm^2} \left[\theta_3 \left(0 \middle| \frac{i\beta^2}{4\pi t} \right) - \theta_4 \left(0 \middle| \frac{i\beta^2}{4\pi t} \right) \right], \quad (5)$$

where $\theta_3(x|y)$ and $\theta_4(x|y)$ are two of the Jacobi elliptic theta-functions and ε acts as an analytic regularization parameter and we will take the limit $\varepsilon \rightarrow 0$ at the end of the calculations.

In order to obtain the free energy in the case of the super-string, one may simply observe that the mass m becomes an operator M . As a consequence, the expression for the statistical sum (5) takes the form

$$F(\beta) = -\frac{1}{2} \int_0^\infty dt t^{\varepsilon-1} (4\pi t)^{-5} \text{Tr} e^{-tM^2} \left[\theta_3 \left(0 \middle| \frac{i\beta^2}{4\pi t} \right) - \theta_4 \left(0 \middle| \frac{i\beta^2}{4\pi t} \right) \right] \quad (6)$$

where M^2 is the mass operator. For the closed superstring in the light cone gauge ($T = \frac{1}{\pi}$) one has (see [13])

$$M^2 = 4 \sum_{i=1}^8 \sum_{n=1}^\infty n \left(N_{n_i}^b + N_{n_i}^f + \tilde{N}_{n_i}^b + \tilde{N}_{n_i}^f \right). \quad (7)$$

Furthermore in this case (closed superstring) we also have to take into account the following constraint on the states of the system [13]

$$\sum_{i=1}^8 \sum_{n=1}^\infty n \left(N_{n_i}^b + N_{n_i}^f - \tilde{N}_{n_i}^b - \tilde{N}_{n_i}^f \right) = 0 \quad (8)$$

This constraint, which reflects the absence of a preferred point on the closed string, may be implemented via the usual identity (see for example [5]). Therefore the trace of the "heat-kernel" of the mass operator becomes

$$\begin{aligned} \text{Tr} \left\{ e^{-M^2 t} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 e^{2\pi i \tau_1 (N - \tilde{N})} \right\} &= 64 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \prod_{n=1}^\infty \left| \frac{1 - e^{2\pi i \tau_1 n}}{1 + e^{2\pi i \tau_1 n}} \right|^{-16} \\ &= 64 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 |\theta_4(0|2\tau)|^{-16} \end{aligned} \quad (9)$$

where $\tau = \tau_1 + i\tau_2 = \tau_1 + i\frac{2t}{\pi}$ and the presence of the factor 64 is due to the degeneracy of the ground states.

Using the asymptotic behavior of the θ_4 for $\tau_1 = 0$ and $\tau_2 \rightarrow 0$, the integrand of Eq. (6) behaves, in this limit, as

$$\theta_4(0|2\tau)^{-16} \Big|_{\tau_1=0, \tau_2 \rightarrow 0} = \frac{\tau_2^8}{2^8} \left(e^{\frac{2\pi}{\tau_2}} - 16e^{\frac{\pi}{\tau_2}} + 120 + O\left(e^{-\frac{\pi}{\tau_2}}\right) \right) \quad (10)$$

Thus, in order to isolate the high-temperature behaviour we may rewrite identically

$$\begin{aligned} \text{Tr } e^{-M^2 t} = 64 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \left[|\theta_4(0|2\tau)|^{-16} - \frac{\tau_2^8}{2^8} \left(e^{\frac{2\pi}{\tau_2}} - 16e^{\frac{\pi}{\tau_2}} + 120 \right) \right] \\ + \frac{\tau_2^8}{2^8} \left(e^{\frac{2\pi}{\tau_2}} - 16e^{\frac{\pi}{\tau_2}} + 120 \right). \end{aligned} \quad (11)$$

From the expression (11) and with the help of the Poisson summation formula, the asymptotic behavior at high temperature of the statistical sum (6) becomes

$$\begin{aligned} F(\beta) \simeq & -\frac{1}{8\pi^7} \left(\frac{\ln 2}{3!} + G(1) \frac{2\pi}{\beta} \right) \\ & - \frac{4}{2^8 \pi^{10}} \int_0^\infty d\tau_2 \tau_2^{\varepsilon+2} \left(e^{\frac{2\pi}{\tau_2}} - 16e^{\frac{\pi}{\tau_2}} + 120 \right) \sum_{n=0}^\infty \exp \left(-\frac{(2n+1)^2 \beta^2}{2\pi \tau_2} \right) \end{aligned} \quad (12)$$

where we have neglected those terms that are exponentially small at high temperature ($\beta \rightarrow 0$) and have put

$$G(s) = \frac{32}{\pi^3 \sqrt{2\pi}} \int_0^\infty d\tau_2 \tau_2^{\frac{s}{2}-6} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 |\theta_4(0|2\tau)|^{-16} - \frac{\tau_2^8}{2^8} \left(e^{\frac{2\pi}{\tau_2}} - 16e^{\frac{\pi}{\tau_2}} + 120 \right) \right]. \quad (13)$$

As a result $G(1)$ is a finite number. The second term of (12) is the one which interests us, since it describes the Hagedorn sector. In fact it is easy to see that the integral converges as soon as $\beta^2 > 4\pi^2$, 2π being the Hagedorn critical temperature: this is a well known result (see for example Ref. [14]). However, our aim is to find an analytical continuation of this expression. To evaluate it, we expand $e^{\frac{2\pi}{\tau_2}}$ and $e^{\frac{\pi}{\tau_2}}$ in series of powers and then we integrate terms by terms. The integrals thus obtained, have to be analytically regularized, and they can be evaluated with the help of the relation

$$\int_0^\infty dy y^{k-4+\varepsilon} \exp \left(-\frac{(2n+1)^2 \beta^2 y}{2\pi} \right) = \Gamma(k-3+\varepsilon) \left(\frac{(2n+1)^2 \beta^2}{2\pi} \right)^{3-k-\varepsilon} \quad (14)$$

which has to be interpreted in the sense of analytic continuation. Moreover the regularization of the integral $\int_0^\infty t^\lambda dt$ as analytical function of λ gives $\int_0^\infty t^\lambda dt = 0$ [15], and therefore the last integral in the second term of (12) gives no contribution to the statistical sum. The sum over n gives the series representation of the ζ -function. The removal of the cutoff is delicate. Again the analytical continuation must be invoked. The first four terms in the sum are treated by making use of functional equation for the Riemann ζ -function, namely

$$\Gamma(z)\zeta(2z) = \Gamma\left(\frac{1}{2} - z\right)\zeta(1-2z)\pi^{2z-1}. \quad (15)$$

As a result, by taking the limit for ε that goes to zero we end up with a representation for the asymptotic behavior at high temperature of the free energy density in terms of the Laurent series

$$F(\beta) \simeq -\frac{1}{8\pi^7} \left[\frac{\ln 2}{3!} + G(1)x + \sum_{k=0}^2 A(k)x^{2k-6} + \sum_{k=1}^\infty B(k)x^{2k} \right] \quad (16)$$

where $x = \frac{2\pi}{\beta}$ and

$$\begin{aligned} A(k) &= \frac{\pi^{2k-7}}{k!} (1 - 2^{6-2k}) \Gamma\left(\frac{7}{2} - k\right) \zeta(7 - 2k) \\ B(k) &= (1 - 2^{1-k})(1 - 2^{-2k}) \zeta(2k) \frac{\Gamma(k)}{(k+3)!}. \end{aligned} \quad (17)$$

One may derive this Laurent representation starting from the Mellin-Barnes representation along the lines of Refs. [11,12]. Here we have presented another derivation.

As we have already mentioned, this series representation characterizes the critical temperature. In fact the series converges when $\beta \geq \beta_c = 2\pi$, β_c being the Hagedorn temperature. It is interesting to notice that, unlike the case of the open superstring, the free energy of the closed superstring is finite for $\beta \rightarrow \beta_c + 0$ (see for example the review papers [5] and [14]).

Inside the radius of convergence, it is possible to evaluate the sum of the series $\sum_{k=1}^{\infty} B(k)x^{2k}$ in terms of known functions. The calculation is tedious but straightforward and we only sketch the derivation. The starting point is the relation (here $x < 1$)

$$\sum_{k=1}^{\infty} (1 - 2^{-2k}) \zeta(2k) \frac{x^{2k+1}}{2k} = -\frac{x}{2} \ln \cos\left(\frac{\pi x}{2}\right). \quad (18)$$

Integrating terms by terms one gets

$$\sum_{k=1}^{\infty} (1 - 2^{-2k}) \zeta(2k) \frac{x^{2k+2}}{2k(2k+2)} = -\frac{1}{2} \int_0^x dt t \ln \cos\left(\frac{\pi t}{2}\right). \quad (19)$$

Now one has [16]

$$\int dt \ln \cos t = \frac{i}{2} Li_2(-e^{2it}) - t \ln 2 - \frac{i}{2} t^2 \quad (20)$$

where $Li_s(z)$ is the polylogarithm function of order s . It is an analytic function for $s, z \in \mathcal{C}$ and $|z| < 1$, defined by the Dirichelet series (see Ref. [17], where this function has been investigated in detail)

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \quad |z| < 1 \quad s \in \mathcal{C}. \quad (21)$$

For $s \in \mathbb{N}$ $s \geq 2$ the function is also defined for $z = 1$ and in this case it is equal to the Riemann ζ -function. Making use of the following property of the polylogarithm function

$$\int_0^y Li_s(-e^{iat}) dt = \frac{1}{ia} [Li_{s+1}(-e^{ia y}) - Li_{s+1}(-1)] \quad (22)$$

and integrating twice terms by terms, one arrives at the final result, which reads

$$\begin{aligned} \sum_{k=1}^{\infty} B(k)x^{2k} &= \frac{4i}{\pi^2 x^3} \tilde{Li}_4(-e^{i\pi x}) - \frac{24}{\pi^3 x^4} \tilde{Li}_5(-e^{i\pi x}) - \frac{60i}{\pi^4 x^5} \tilde{Li}_6(-e^{i\pi x}) \\ &+ \frac{60}{\pi^5 x^6} \tilde{Li}_7(-e^{i\pi x}) - \frac{60}{\pi^5 x^6} \tilde{Li}_7(-1) - \frac{6}{\pi^3 x^4} \tilde{Li}_5(-1) - \frac{1}{2\pi x^2} \tilde{Li}_3(-1) \\ &+ \frac{4\pi}{48} \ln 2 + \frac{16i\pi^2}{840} x(1 - 2^{-\frac{1}{2}}) \end{aligned} \quad (23)$$

where we have introduced the functions

$$\tilde{Li}_n(-e^{i\pi x}) = Li_n(-e^{i\pi x}) - 2^{\frac{n+1}{2}} Li_n(-e^{i\pi \frac{x}{\sqrt{2}}}) \quad (24)$$

Let us conclude with some remarks. Eq. (23) has been obtained for $x \leq 1$. However, the function $Li_s(z)$ admits an analytical continuation in the cut z -plane, the cut being imposed from 1 to $+\infty$ along the positive real axis [17]. A direct calculation shows that the function (23) is real for $x \leq 1$. For $x > 1$, namely for a temperature bigger than the Hagedorn one, the function is finite but it acquires a non-vanishing imaginary part. The idea to use the analytical continuation in string theory is not new. It can be found for example, in Ref. [18], where the appearance of imaginary terms in the free energy above the Hagedorn temperature has been pointed out. Note, however, that the use of the analytical continuation may present some problems (see the criticism contained in Ref. [19]).

Having the explicit form of the statistical sum, the imaginary part can be computed and the result is

$$\text{Im } F_\beta \equiv \frac{\pi^2}{196} \left[\theta(x-1)(x^2-1)^3 - \theta(x-\sqrt{2}) \frac{(x^2-2)^3}{4} \right] \quad (25)$$

This is not surprising, because naive considerations based on the exponential dependence on the mass of the state level density leads to the conclusion that above the critical temperature the one-loop partition function is diverging. Since the analytical continuation of the free energy has a non-vanishing imaginary part, in analogy with field theoretical case [20] and finite temperature one-loop quantum gravity [21], one might conclude that the "effective potential" for superstring, above the Hagedorn temperature, develops a kind of new "local minimum" and there might be a decay from the old vacuum to the new one [18], the decay rate of this "tunneling" being described by the imaginary part we have computed. One might conclude that above the Hagedorn temperature, the superstring would become a metastable system. However, it should be noted that we are neglecting gravitational effects and this could not be justified when the temperature approaches the Hagedorn one [13]. Finally similar considerations may be done for the heterotic string [22] and we hope to return on this issue elsewhere.

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